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Qudit surface codes and gauge theory with finite cyclic groups

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Abstract

Surface codes describe quantum memory stored as a global property of interacting spins on a surface. The state space is fixed by a complete set of quasi-local stabilizer operators and the code dimension depends on the first homology group of the surface complex. These code states can be actively stabilized by measurements or, alternatively, can be prepared by cooling to the ground subspace of a quasi-local spin Hamiltonian. In the case of spin-1/2 (qubit) lattices, such ground states have been proposed as topologically protected memory for qubits. We extend these constructions to lattices or more generally cell complexes with qudits, either of prime level or of level d^ℓ for d prime and $\ell \geqslant 0$, and therefore under tensor decomposition, to arbitrary finite levels. The Hamiltonian describes an exact $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge theory whose excitations correspond to Abelian anyons. We provide protocols for qudit storage and retrieval and propose an interferometric verification of topological order by measuring quasi-particle statistics.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

There is a rich history to the study of topologically ordered states of matter. Such states are defined by the property that all physical correlation functions are topological invariants. In the field of condensed matter, these states have been proposed as ground states of models for high temperature superconductors and for fractional quantum Hall states [20]. Furthermore, it has been demonstrated that such order can arise as a low energy property of hard core bosonic spin lattice models. In contrast to the familiar situation with spontaneous symmetry

breaking, here the ground states exhibit more symmetry than the microscopic equations of motion. It has been suggested that such emergent properties may model gauge fields and particles found in nature [15]. In the field of quantum information it was shown by Kitaev [12] that ground states of Hamiltonians which can be expressed as a sum over quasi-local stabilizer operators provide for topologically protected qubit memories. These states are referred to as surface codes. They are robust to arbitrary quasi-local perturbations and have Abelian anyonic excitations. In order to perform universal fault-tolerant quantum processing, it is necessary to use non-Abelian anyonic excitations that transform under an appropriate group [12]. From the algorithmic point of view attempts have been made to understand quantum computing in terms of non-Abelian anyon operations [1]. The pursuit of microscopic lattice models that admit anyonic eigenspaces sufficient for universal quantum computation is an area of active research [7].

This survey attempts to exhaust the topic of surface codes for topologically protected qudit memories. While not as powerful as fault tolerant models with non-Abelian anyons, these models offer a new perspective on non-local encoding of quantum information and give us insight into microscopic realizations of lattice gauge theories. Surface codes for two level systems [12] are by now well understood. Their implications for error-resistant quantum computer memories have also been considered [5]. In the error-correction context, the topologically ordered eigenstates may be understood as a particular case of quantum stabilizer codes (e.g. [8]). The error lengths of the resulting stabilizer codes are not exceptional, and only rarely do anyonic systems appear in classifications of near-optimal quantum codes. (Optimality in this sense refers to minimizing the number of code-qubits against the number of errors a code may correct.) Yet all the error correction operations are local upon the lattice in which the quantum data are stored, which might improve scalability. Moreover, an aside to an argument focused on deriving a famous stabilizer code from the topology of the real projective space in fact demonstrates that a qubit lattice is not required [6]. Rather, a two-complex (see, e.g., [14]) suffices, where a two-complex is a generalization of a graph in which discs are also allowed with edge boundaries. On the physical system which places a qubit on each edge of a (cellular or simplicial) two-complex Γ, there exists a Hamiltonian whose topologically ordered (stabilizer-code) ground states are parametrized by the first homology group of the complex with bit coefficients: $H_1(\Gamma, \mathbb{F}_2)$. The Hamiltonian is a sum of vertex and edge terms which are proportional to either tensors of Pauli Z operators around qubits on edges adjacent to a vertex or tensors of X operators on edges bounding a face of the complex.

For some time the existence of stabilizer codes over qudits (d prime) has been known [8]. Yet only recently have results on the topic become as strong as those applicable in the bit case, including estimates of optimal code lengths etc ([11], see also [9]). Moreover, extensions to prime-power (d^ℓ) level qudits (actually qud^ℓ its) have also been found, so that tensors provide a stabilizer formalism for all finite-level systems. In this work, we exploit the new stabilizer formalism to construct codes on a two-complex whose edges carry prime-d-level qudits, and we also outline the extension to d^ℓ-level qudits. The associated ground states are parametrized by $H_1(\Gamma, \mathbb{F}_{d^\ell})$, where the coefficient field is viewed as an Abelian group under addition. This requires a few new ideas, although care must be taken with sign conventions which were vacuous in the earlier work on $\mathbb{F}_2$-coefficients. Thus, after tensoring we have constructed surface codes with qudits for arbitrary finite d placed on the edges of a generic orientable two-complex Γ. Recent work by Bombin and Martin-Delgado [2] investigates classical and quantum homological error correction codes. They construct a class of surface codes for qudits which asymptotically saturates the maximum coding rate and provide several example encodings on various two-complexes. Here we do not address the issue of coding efficiency. Rather we concentrate on explicit constructions of Hamiltonians that support qudit surface

codes in their ground eigenstates and describe how one might encode and decode therein. A second difference is the choice of the definition of stabilizer codes for composite d. This work views d as a product of prime powers $p^\ell = q$ and exploits Pauli X and Z operators and Fourier transforms relevant to these fields, following [11]. This creates a topological code distinct from a stabilizer code using the usual X and Z operators for d-level systems, following [9]. Bombin and Martin-Delgado follow the conventions of [9].

The manuscript is intended to be self-contained. Thus, section 2 opens by reviewing some the required facts on stabilizer codes. In order to aid readers less interested in the general case, section 3 treats prime-d level encoding on surfaces separately. Methods for encoding, decoding and stabilizer measurements are given in section 4. Extensions to the case of prime power qudit encodings are given in section 5. Errors in our model correspond to low lying excitations in the Hamiltonian whose superselection sectors may be viewed as massive particles on the underlying cellulation. In section 6, it is shown that our model reproduces a $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge theory where errors are described by particle–antiparticle pairs of charge/flux dyons. We propose an interferometer circuit for measuring the statistics of these quasiparticles. We conclude with a summary of the main results.

2. Qudit stabilizer codes

We next review stabilizer codes [8, 11]. This section focuses on the case of qudits with a prime number of levels. The first subsection recalls the definition and a basic technique. The next subsection generalizes a well-known construction from bits to dits.

2.1. Stabilizers and ground states

Let d be a prime number, and consider the qudit state space $\mathcal{H}(1,d) = \mathbb{C}|0\rangle \oplus \cdots \oplus \mathbb{C}|d-1\rangle$, with a pure state of n qubits being a ket within $\mathcal{H}(n,d) = \mathcal{H}(1,d)^{\otimes n}$. A possible generalization of the Pauli operators on $\mathcal{H}(1,d)$ would be to consider the group generated by the following unitary matrices:

$$\begin{aligned} X|j\rangle &= |j+1 \bmod d\rangle \\ Z|j\rangle &= \xi^j|j\rangle, \qquad \text{for} \quad \xi = \exp(2\pi \mathrm{i}/d). \end{aligned} \tag{1}$$

These are not Hermitian unless $d = 2$. The qudit Pauli-tensor group, say $\mathcal{P}(n,d) \subsetneq U[\mathcal{H}(n,d)]$, is the group of unitary matrices generated by n-fold tensors of elements of $\{I_d, X, Z\}$.

We might be more explicit in the description of $\mathcal{P}(n,d)$. First, for $n = 1$, label the multiplication in $\mathbb{F}_d$ to be a dot-product. Then $Z^b X^a = \xi^{a\bullet b} X^a Z^b$. More generally, for dit-strings $a, b \in (\mathbb{F}_d)^n$, we use $X^{\otimes a}$ and $Z^{\otimes b}$ to abbreviate $X^{a_1} \otimes X^{a_2} \otimes \cdots \otimes X^{a_n}$ and similarly $Z^{\otimes b}$ for $Z^{b_1} \otimes Z^{b_2} \otimes \cdots \otimes Z^{b_n}$. For the n-entry dot-product with values in $\mathbb{F}_d$, we have $Z^{\otimes b} X^{\otimes a} = \xi^{a\bullet b} X^{\otimes a} Z^{\otimes b}$. Thus explicitly

$$\mathcal{P}(n,d) = \{\xi^c X^{\otimes a} Z^{\otimes b};\, a, b \in (\mathbb{F}_d)^n, c \in \mathbb{F}_d\} \tag{2}$$

The qudit stabilizer groups are subgroups $G \subseteq \mathcal{P}(n,d)$. The code subspace of such a stabilizer group is the joint +1 eigenspace of all $g \in G$.

Of course, such joint eigenspaces might well be trivial. Yet a standard argument shows that they are nontrivial in certain cases. This technique is so fundamental to stabilizer code manipulation that we wish to highlight it; it will be used several more times in the course of the work. While actually an elementary technique from representation theory, it has also featured prominently in the quantum computing literature [13].

Stabilizer code projectors. The sum of unitary maps $\pi = (\#G)^{-1}\sum_{g\in G} g$ is a projection onto the code-subspace. We present the argument. First, $\pi^2 = \pi$ since $\pi g = \pi$ for any $g \in G$. Second, $\pi = \pi^\dagger$ since adjoints are inverses in the unitary group. Thus π is a projection, and it remains to verify that π projects onto the stabilizer code space. Now split $\mathcal{H}(n,d) = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$ into irreducible orthogonal unitary subrepresentations of G. For each V_j, the image under π and its orthogonal complement form a decomposition of V_j. Thus by irreducibility, π either preserves a V_j or $\pi V_j = 0$. Clearly the former holds for any irrep (i.e. irreducible representation) within the code subspace of G. On the other hand, if $\langle\psi|g|\psi\rangle \neq 1$ for some g, then the latter holds.

As a remark, irreps within the code subspace of G must be one-dimensional and are also known as *trivial representations*. As a second remark, the code subspace is nonzero iff $\mathrm{Trace}(\pi) \neq 0$ iff $(G \cap \{\xi^j I_d\}) = \{I_{d^n}\}$.

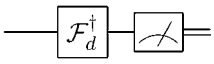
In the Hermitian case ($d = 2$), it is standard that all eigenvalues of group elements are ± 1, so that a suitable Hamiltonian for which the code space is the ground state is $-\pi$. For general d, the eigenvalues lie within the unit circle, so that -1 is still the least possible real part. Also, $g^\dagger|\lambda\rangle = (1/\lambda)|\lambda\rangle = \overline{\lambda}|\lambda\rangle$ since $g^\dagger = g^{-1}$. Thus, one may place the qudit code subspace into the ground state of a Hamiltonian by adjusting each summand of π with a Hermitian conjugate: $H = \sum_{g\in G} -(g + g^\dagger)$, so that the eigenvalues of the summands are then $-2\mathrm{Re}[\mathrm{spec}(g)]$.

2.2. Quantum circuits for qudit stabilizer measurements

Given an n-qudit system, it is important for purposes of error correction to be able to test whether or not a state $|\psi\rangle$ lies within the stabilizer code of some $G = \langle\{g_j\}\rangle \subseteq \mathcal{P}(n,d)$. It suffices to test whether $|\psi\rangle$ is a +1 eigenvector of each generator g_j. We sketch quantum circuits which achieve such a measurement.

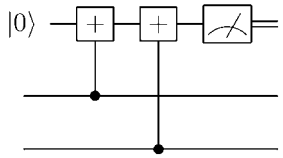
Let $\mathcal{F}_d = d^{-1/2}\sum_{j,k=0}^{d-1} \xi^{jk}|j\rangle\langle k|$ be the qudit Fourier transform. Considering eigenkets, $\mathcal{F}_d^\dagger X \mathcal{F}_d = Z$. Now the number operator $\mathbf{n} = \sum_{j=0}^{d-1} j|j\rangle\langle j|$ suffices to infer the eigenvalue of Z and project into the appropriate eigenstate. As a circuit, we might denote a number operator measurement with the Z symbol, one of several common conventions in the qubit case:

Determination of the X eigenstate may be accomplished by



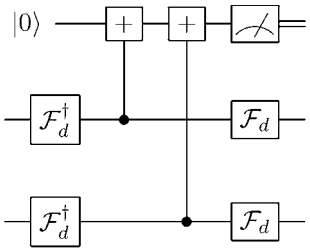
Similarly, there is some one-qudit unitary which will diagonalize any $X^a Z^b \in \mathcal{P}(1,d)$, usually not a Fourier transform. Yet using the diagonalization and a number operator one may infer an eigenstate.

For $Z^{\otimes k}$ and $X^{\otimes k}$, we suggest using addition gates along with a qudit ancilla. We will denote $|j,k\rangle \mapsto |j,(j+k) \bmod d\rangle$ by a typical control bullet with the target (in the formula second) line holding a + gate. The the following construction of $Z^{\otimes 2}$ generalizes for $Z^{\otimes k}$:



For $Z^{\otimes k}|j_1, j_2, \ldots, j_k\rangle = \xi^{j_1+\cdots+j_k}|j_1, j_2, \ldots, j_k\rangle$, and we have placed $|(j_1 + \cdots + j_k) \bmod d\rangle$ on the ancilla line before the number operator is applied. Note that $Z \otimes Z^{-1}$ results by

replacing one of the modular addition gates above with modular subtraction. Powers of operators are measured by multiple applications of the sum gate appropriately. Finally, $\left(\mathcal{F}_d^\dagger\right)^{\otimes k} X^{\otimes k} \mathcal{F}_d^{\otimes k} = Z^{\otimes k}$, so that the following diagram for $X^{\otimes 2}$ extends:



Using similarity transforms by qudit Fourier transforms, we may similarly achieve $X \otimes Z \otimes Z \otimes X$ etc. Yet more generally, the comment on existence of diagonalizations above produces circuits for arbitrary elements $g \in \mathcal{P}(n, d)$.

3. Homologically ordered ground states for prime qudits

It is typical to place topological orders on explicit planar or spatial lattices of spin-j particles, e.g. square, triangular, hexagonal, Kagome, etc. An alternative was presented in Freedman and Meyer's derivation of certain error-correcting codes of Shor and LaFlamme [6]. Namely, qubits could be placed on the edges of a two-complex Γ, and an appropriate Hamiltonian would have the dimension of its degenerate ground-state eigenspace equal to the number of classes within $H_1(\Gamma, \mathbb{F}_2)$. We next extend this construction to prime-level qudits; the task is mainly to keep track of sign conventions which are vacuous in $\mathbb{F}_2$. We then check that the ground-state eigenspace is similarly spanned by kets associated to elements of $H_1(\Gamma, \mathbb{F}_d)$, by applying stabilizer-code techniques.

3.1. Cellular Hamiltonians

Label $\mathcal{V}$ to be the vertices of $\Gamma, \mathcal{E}$ to be the edges, and $\mathcal{F}$ to be the faces. We also require properties that hold if Γ is a cellulation of an orientable, compact, connected surface. Specifically, each edge has a boundary of exactly two vertices and each face has an orientation according to which each edge lies in the boundary of two faces with the edge taking opposite orientations in the boundary of each face. Finally, Γ is finite and $H_2(\Gamma, \mathbb{F}_d)$ is a copy of $\mathbb{F}_d$ spanned by $[\Gamma]$, the sum of all faces with their orientation according to Γ.

We briefly review the appropriate homology. Label the chain sets to be formal sums of vertices, edges and faces respectively: $C_0(\Gamma, \mathbb{F}_d) = \mathrm{span}_{\mathbb{F}_d}(\mathcal{V})$, $C_1(\Gamma, \mathbb{F}_d) = \mathrm{span}_{\mathbb{F}_d}(\mathcal{E})$ and $C_2(\Gamma, \mathbb{F}_d) = \mathrm{span}_{\mathbb{F}_d}(\mathcal{F})$. We generally drop the Γ and coefficient system, which should be clear from the context. Since Γ is a cell complex, there exist boundary operators

$$C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \tag{3}$$

with $\partial^2 = 0$ [14]. For example, if an edge e connects v_1 and v_2, say $e = [v_1, v_2]$, then $\partial e = v_1 - v_2 = v_1 + (d-1)v_2$. Note that the definition of Γ demands that edges $e \in \mathcal{E}$ are images of $[0, 1]$ within Γ, and hence all edges are implicitly *oriented*. The coefficients further allow for $\mathbb{F}_d$-valued multiplicities on each edge. Since $\partial^2 = 0$, we have $\ker(\partial_1) \supseteq \mathrm{image}(\partial_2)$ for $\partial_j : C_j \to C_{j-1}$. Thus we may define the $\mathbb{F}_d$ vector space $H_1(\Gamma, \mathbb{F}_d) = \ker(\partial_1)/\mathrm{image}(\partial_2)$. This first homology group is well known to be a topological invariant, i.e. any topological space homotopic to that underlying Γ will produce an $H_1(\Gamma, \mathbb{F}_d)$ of the same dimension.

Homology elements are represented by cycles, i.e. elements of the kernel of the boundary operator. However, several elements might represent the same class, differing by a boundary, i.e. an element of $\partial(C_2)$. For $\omega \in C_1$, we use $[\omega]$ to denote the equivalence class $\omega + \text{image}(\partial_2)$, which is an element of $H_1(\Gamma, \mathbb{F}_d)$.

Recall that any Hamiltonian on n-qudits may be written as a sum of tensor products of Hamiltonians (Hermitian matrices) on each factor. The degree of a summand in the tensor basis is the greatest number of non-identity factors in any term. A k-local Hamiltonian is a Hamiltonian whose degree is bounded by k in some decomposition. The topologically ordered Hamiltonians defined below are k-local for k the maximum of the valence of any vertex and the number of edges on any face.

Let $n = \#\mathcal{E}$, and consider placing a qudit on each $e \in \mathcal{E}$. Again, each edge is the image of $[0, 1]$ and is oriented (by Γ) from one vertex to the other. For the qudits associated with each edge, the $|1\rangle$ excitation of the edge will be implicitly associated to this orientation, while the $|d-1\rangle$ state corresponds to the other.

On the associated physical system $\mathcal{H}(n, d)$, let X_e and Z_e denote the operator applied to the qudit of that edge with identity operators buffered into the remainder of the tensor. For each $v \in \mathcal{V}$, we define a Pauli tensor and vertex Hamiltonian by

$$\begin{aligned} g_v &= \prod_{e=[*,v]} Z_e \prod_{e=[v,*]} Z_e^{-1} \\ H_v &= -\left(g_v + g_v^\dagger\right). \end{aligned} \tag{4}$$

For some $U > 0$, we then define the potential energy term of a topologically ordered Hamiltonian by $H_\partial = U \sum_{v\in\mathcal{V}} H_v$.

The notation H_∂ has been chosen for the following reason. Suppose that $\omega = \sum_{e\in\mathcal{E}} n_e e$ is a chain, with each $n_e \in \mathbb{F}_d$. There is an associated qudit computational basis state, say $|\omega\rangle$, which is local and places the qudit of each e in state $|n_e\rangle$. We claim that $|\omega\rangle$ is a ground state of H_∂ iff $\partial\omega = 0$, i.e. ω is a cycle. To see this, one verifies that $g_v|\omega\rangle = \xi^c|\omega\rangle$ where $\partial\omega = cv + \sum_{w\neq v} c_w w$. Hence $|\omega\rangle$ is in the stabilizer $\langle\{g_v\}\rangle \subseteq \mathcal{P}(n, d)$ iff $|\omega\rangle$ is an eigenstate of each H_v of minimial (real) eigenvalue iff $|\omega\rangle$ is in the degenerate ground-state eigenspace of H_∂.

Strictly speaking, one should not refer to the ground state of H_∂ as being topologically ordered. Admittedly, ground states are of the form $|\psi_g\rangle = \sum \alpha_\omega|\omega\rangle$ for ω a cycle, colloquially a loop of excited edges. For $d > 2$, the edges must be properly oriented, and hitting every edge of a Y junction is allowed if multiplicities are accounted for. Yet the cycle subspace is not a topological invariant. Indeed, should Γ be a cell complex, subdividing Γ by breaking each 2-simplex (triangle) into several subtriangles will generally increase the size of $\ker(\partial_1)$, although such a subdivision does not change the topology of the underlying manifold. Thus, we next add a kinetic energy term to the potential, splitting the degeneracy of H_∂ and reducing to a final ground state capturing homology.

For each face f, the face Hamiltonian H_f is defined as follows. Orient f according to the orientation of the manifold underlying Γ. Label edges by $\partial f = \sum_{k=1}^{p} o_k e_k$ for $o_k \in \{1, d-1\}$. Then we define

$$g_f = X_{e_1}^{o_1} X_{e_2}^{o_2} X_{e_3}^{o_3} \ldots X_{e_p}^{o_p} \qquad H_f = -\left(g_f + g_f^\dagger\right). \tag{5}$$

With these choices, $[H_f, H_v] = 0$ for all faces f and vertices v. For the two edges incident on a given vertex will be in the boundary of some face, and after correcting for orientation conventions this commutativity check reduces to $[X \otimes X, Z \otimes Z^{-1}] = 0$ (see figure 1). Hence, for some constant $h > 0$, we might define $H_{\text{KE}} = h \sum_{f\in\mathcal{F}} H_f$. Due to commutativity, the

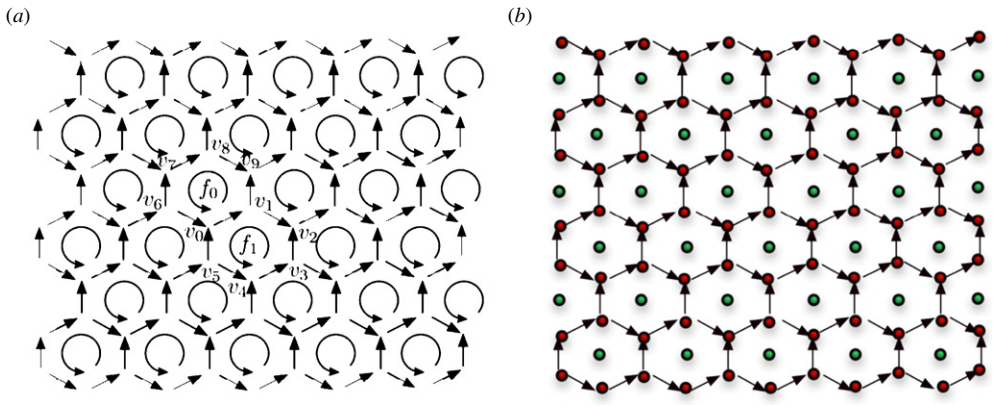


Figure 1. Cellulation of an orientable surface. Each system particle (qudit) is represented by an edge. Particle interactions occur between all edges that meet at a common vertex and all edges comprising a plaquette boundary. (*a*) In this example, physical qudits reside on the vertices of a Kagome' lattice such that the resultant cellulation is a honeycomb lattice. Edge and face orientations are indicated. For the vertices v_0, v_1 and faces f_0, f_1 the mutually commuting operators in the Hamiltonian are $g_{v_0} = Z_{[v_6,v_0]}Z_{[v_5,v_0]}Z^{-1}_{[v_0,v_1]}$, $g_{v_1} = Z_{[v_0,v_1]}Z^{-1}_{[v_1,v_9]}Z^{-1}_{[v_1,v_2]}$, $g_{f_0} = X_{[v_0,v_1]}X_{[v_1,v_9]}X^{-1}_{[v_8,v_9]}X^{-1}_{[v_7,v_8]}X^{-1}_{[v_6,v_7]}X_{[v_6,v_0]}$, $g_{f_1} = X^{-1}_{[v_0,v_1]}X^{-1}_{[v_1,v_2]}X_{[v_3,v_2]}X_{[v_4,v_3]}X_{[v_5,v_4]}X^{-1}_{[v_5,v_0]}$. (*b*) Same cellulation with vertex (red) ancilla and face (green) ancilla. These can be used to perform local stabilizer checks or to mediate many body interactions between edges from physical 2-local interactions as described in section 4.3.

kinetic energy Hamiltonian respects the ground-state degeneracy of H_∂. Label $H = H_\partial + H_{\mathrm{KE}}$. We next show that the dimension of the ground-state degeneracy of total Hamiltonian

$$H = H_\partial + H_{\mathrm{KE}} \tag{6}$$

(over $\mathbb{C}$) corresponds to the number of elements of $H_1(\Gamma, \mathbb{F}_d)$.

Before the argument, we present extra notation and sketch what will follow. Three stabilizer groups are relevant, namely $G_v = \langle \{g_v\}_{v\in\mathcal{V}} \rangle$, $G_f = \langle \{g_f\}_{f\in\mathcal{F}} \rangle$ and $G = \langle \{g_v\}_{v\in\mathcal{V}} \cup \{g_f\}_{f\in\mathcal{F}} \rangle$. This notation does not refer to the expectation of an observable but rather the minimal subgroups of $\mathcal{P}(n, d)$ holding each set. The idea behind the proof is then to argue that the composition stabilizer code projectors of G_f and G_v are the stabilizer code projector for G, equivalently the ground state of H.

3.2. Homology class ground states

The goal of this section is to associate the degeneracy (dimension) of this ground state of $H = H_\partial + H_{\mathrm{KE}}$ to $\#H_1(\Gamma, \mathbb{F}_d)$. We accomplish this in two distinct cases for the manifold underlying Γ:

(i) The manifold is orientable, compact and has no boundary, so that $H_2(\Gamma, \mathbb{F}_d) = \mathbb{F}_d$.
(ii) The manifold is compact with boundary and has $H_2(\Gamma, \mathbb{F}_d) = 0$. For homology is a homotopy invariant, and such a surface retracts into its one skeleton.

Assertion. Let $\mathcal{H}_{\mathrm{loop}}$ denote the ground state of H_∂ and $\mathcal{H}_{[\omega]} = \oplus_{\eta\in[\omega]}\mathbb{C}|\eta\rangle$:

$$\mathcal{H}_{\mathrm{loop}} = \oplus_{\omega\in\ker\partial}\mathbb{C}|\omega\rangle = \oplus_{[\omega]\in H_1(\Gamma,\mathbb{F}_d)}\mathcal{H}_{[\omega]}. \tag{7}$$

Throughout this section, let $\pi = \#G^{-1}\sum_{g\in G} g$. Suppose either case i or case ii. Then for each $[\omega]$, the restriction of π to $\mathcal{H}_{[\omega]}$ is a rank-one projector whose (nonzero) image is an element of $\ker(H_\partial + H_{\mathrm{KE}}) = \ker H$.

To verify this, suppose $|\omega\rangle$ is the computational basis state of some cycle $\omega \in C_1$ (i.e. $\partial\omega = 0$). Then we may also speak of $[\omega] \in H_1(\Gamma, \mathbb{F}_d)$, $|\omega\rangle$ is in the ground state of H_∂. Label

$$|[\omega]\rangle \stackrel{\text{def}}{=} \pi_f|\omega\rangle = (\#G)^{-1}\sum_{g\in G} g|\omega\rangle. \tag{8}$$

It suffices for the assertion to show the following.

- If ω_1 and ω_2 each lie in $[\omega]$, then $|[\omega_1]\rangle$ and $|[\omega_2]\rangle$ differ by a global phase.
- If $|\omega\rangle \neq 0$, then $|[\omega]\rangle \neq 0$.

This suffices to see the restriction of π is a rank-one projector, since the first item demands the rank $\leqslant 1$ and the second demands the rank $\geqslant 1$.

We begin with the first item, writing $\omega_1 - \omega_2 = \eta \in \operatorname{im}\partial_2$. Since the underlying manifold of Γ is orientable, suppose for convenience that all faces f have positive orientation. Then for $\eta = \sum_{f\in S(\eta)} f$ we put $g_\eta = \prod_{f\in S(\eta)} g_f$, implying $|\omega_1\rangle = g_\eta|\omega_2\rangle$. Note that $g_\eta\pi_f = \pi_f g_\eta = \pi_f$. Thus $|[\omega_1]\rangle = \pi_f g_\eta|\omega_2\rangle = \pi_f|\omega_2\rangle = |[\omega_2]\rangle$.

We next demonstrate that $\pi_f|_{\mathcal{H}_{[\omega]}}$ has rank $\geqslant 1$. As discussed in section 2, it suffices to show that the trace of this projection, when restricted to the subspace $\mathcal{H}_{[\omega]}$ which it preserves, is nonzero, and that immediately follows if $\xi^\ell I_{d^n} \in G_f$ demands $\xi = 1$. For all other elements of $\mathcal{P}(n, d)$ are traceless when restricted to $\mathcal{H}_{\text{loop}}$, since $g_\eta|\omega\rangle = |\omega + \partial\eta\rangle$. Case i and case ii differ somewhat. In each case, multiples of the identity in G_f are products g_η for $[\eta] \in H_2(\Gamma, \mathbb{F}_d)$. In case i, besides the empty product of the g_f we also produce multiples of I_{d^n} as the full product $\prod_{f\in\mathcal{F}} g_f^k$, $0 \leqslant k \leqslant d-1$. This corresponds to $H_2(\Gamma, \mathbb{F}_d) = \mathbb{F}_d$. Yet for these products $\xi = 1$, as may be verified at an individual edge. In case ii, there is no nontrivial product of the g_f which produces a multiple of the identity. This is due to the retraction demanding $H_2(\Gamma, \mathbb{F}_d) = 0$, the second homology of the one complex we may retract onto. Colloquially, taking a sum of all faces will force a boundary edge to be acted on nontrivially by g_f for the single face it bounds. Thus in case ii the only multiple of the identity is the trivial product of the g_f, and $\xi = 1$ tautologically. In each case, $\pi_f|_{\mathcal{H}_{[\omega]}}$ is not traceless and hence has rank at least 1. Given the last paragraph, the rank is exactly 1.

Retracing the argument above, we may compute the image under π of the code space of G_v is $\oplus_{[\omega]\in H_1(\Gamma,\mathbb{F}_d)}\mathbb{C}|[\omega]\rangle$, which is also the code space of G. Since π_f is a rank-1 projector when restricted to each $\mathcal{H}_{[\omega]}$, we have the following:

$$\dim_{\mathbb{C}}(\text{groundstate of } H) = \#H_1(\Gamma, \mathbb{F}_d). \tag{9}$$

3.3. Ground states on a punctured disc

In practice, constructing physical realizations of Hamiltonians corresponding to two-complexes without boundary is daunting. It is possible to simply identify opposite qudits on the square fundamental domain of $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, but this would require some sort of nonlocal coupling on the boundary in addition to the standard lattice coupling. Given a lattice Hamiltonian that arises from electromagnetic coupling, one could speculate about some kind of apparatus (perhaps involving fibre-optic cabling [19]) which allows for interactions between boundary qudits.

Alternately, we might modify the homological ground states to allow for a surface with a boundary curve and punctures. Consider a cellulation Γ of a disc with k punctures. An example with $k = 2$ is shown in figure 2. Label the jth puncture face f'_j which has the same orientation as Γ. Also label the outer boundary of the disc $\partial\Gamma$ and the boundaries of the jth puncture $\partial f'_j$. Analogous to the previous construction, the Hamiltonian on Γ is defined $H' = H_\partial + H'_{KE}$. Here the kinetic term is modified so that the set of face operators does not include operators on the

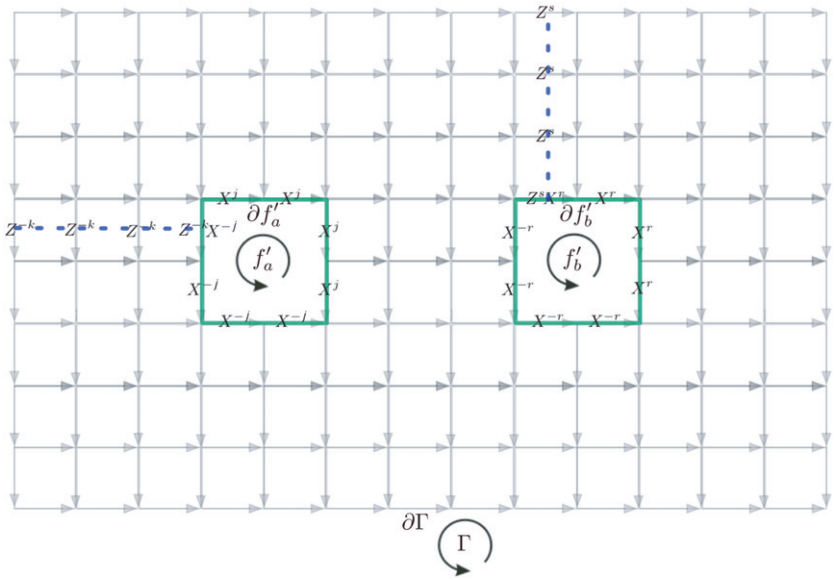


Figure 2. An oriented two-complex Γ, which is a cellulation of a two punctured disc encoding two logical qudits in n physical qudits. Vertex operators H_v are k local where k is the valence of the vertex whereas all face operators H_f are 4 local in this example. Ground states are +1 eigenstates of the stabilizer group G', but not all the stabilizer generators are independent. There are two independent non-trivial cycles on Γ which can be generated by closed loops of X operators around the boundaries $\partial f'_a$ and $\partial f'_b$. Similarly, there are two independent non-trivial cycles on the dual $\bar{\Gamma}$ which can be generated by strings of Z operators that connect two independent pairs of boundaries of the complex. Shown are the Pauli group operations $Z^k X^j$ on qudit a and $Z^s X^r$ on qudit b.

punctured faces f'_j, i.e. $H'_{KE} = h \sum_{f\in\mathcal{F}'} H_f$, where $\mathcal{F}' = \mathcal{F}\backslash\{f \in \cup_{j=1}^{k} f'_j\}$. Consequently, there are edges on the boundaries $\partial f'_j$ that are acted on by X operators from faces on one side only. Another way to see this is that all edges of the dual cellulation that cross the boundary $\partial f'_j$ share a common vertex located at f'_j in Γ. Each edge in Γ has two vertices in $\mathcal{V}$, hence the product over all vertex operators is

$$\prod_{v\in\mathcal{V}} g_v = I_{d^n}. \tag{10}$$

Not every edge in Γ borders two faces in $\mathcal{F}'$, however, and the product over all face operators is:

$$\prod_{f\in\mathcal{F}'} g_f = C_{\partial\Gamma}(X) \prod_{j=1}^{k} C_j(X), \tag{11}$$

where $C_{\partial\Gamma}(X) = \prod_{e_j\in\partial\Gamma} X_{e_j}^{o_j}$ and $C_j(X) = \prod_{e_j\in\partial f'_j} X_{e_j}^{o_j}$. The orientation $o_j = 1$ if the edge e_j is oriented in the same direction as the boundary on which the edge resides, and $e_j = d-1$ if the orientations are opposite.

First we argue that the code space in nonempty. Recall, the code states are defined as +1 eigenstates of the stabilizer group $G' = \langle\{g_f | f \in \mathcal{F}'\} \sqcup \{g_v\}\rangle$. The operators $\langle\{g_v\}\rangle$ and $\langle\{g_f\}\rangle$ commute and the only additional relations obtained from the stabilizer group, embedded in equations (10) and (11), guarantee that $\left(G' \bigcap \xi^j I_{d^n}\right) = I_{d^n}$. We next show that the code space is $\mathcal{H}_{gr} = \mathcal{H}(k, d)$ by considering the action of operators that commute with any member of

G' but act non-trivially on $\mathcal{H}_{gr}$. One such set of operators is non-trivial $\mathbb{F}_d$ valued cycles on Γ generated by $\{C_j(X)\}$. We do not include the non-trivial cycles generated by $C_{\partial\Gamma}(X)$ because by equation (11) their action on the code subspace is not independent but can be generated by the cycles around the boundaries of the punctures. A non-trivial cycle on $\tilde{\Gamma}$ is generated by a string of Z operations along a path $Path(j)$ that begins on an edge of $\partial f'_j$ and ends on an edge of $\partial\Gamma$ without touching other edges on puncture boundaries. We denote the generator of such a cycle $C_j(Z) = \prod_{e_j \in Path(j)} Z_{e_j}^{o_j}$ where $o_k = 1$ at the edge $e_k \in \partial f'_j$ if e_k and $\partial f'_j$ share the same orientation and $o_k = d-1$ otherwise. The other o_j are chosen in a consistent way such that $[H', C(Z)_{f'_j}] = 0$. The operators on cycles satisfy the commutation relations $C(Z)_j^a C(X)_j^b = \xi^{ab} C_j(X)^b C_j(Z)^a$ for $a, b \in \mathbb{F}_d$ as is easily verified by considering the action on the one intersecting edge $e \in \partial f'_j$. As such the set $R_j = \{C_j(Z)^a C_j(X)^b)\}_{a,b=0}^{d-1}$ generates a representation of the Pauli group $\mathcal{P}(1,d)$. For sufficiently spaced punctures, all paths $Path(j)$ exist and the group $R = \langle \{R_j\}_{j=1}^k \rangle$ forms a representation of $\mathcal{P}(k,d)$. We then find that the ground subspace of H' encodes k qudits and the set R_j performs local Pauli group operations on the jth qudit.

In a lattice implementation of our model Hamiltonian, the punctures may arise as physical defects in the system. Coding operations that correspond to cycles around defects vividly illustrate the fact that even short-ranged correlators (short relative to the system size) in a topologically ordered state can have non-trivial values.

4. Quantum memory: input/output and error detection

We next describe how one might exploit Abelian anyons as quantum memories; the qubit case has been studied thoroughly [5]. In the new setting of prime level qudits, we must treat storage and retrieval of quantum data. It is also possible to generalize earlier discussions of stabilizer operations on topologically stored data while in code, but we will not treat that topic here.

4.1. Storing qudits

Placing quantum data into such a $|[\omega]\rangle$ is difficult. For large lattices, this would be a special case of the qudit state-synthesis problem. Universal circuits of two-qudit operators capable of reaching arbitrary n qudit states are known to scale exponentially with the number of qudits [3]. In this section, we propose an alternative which requires a number of stabilizer measurements that is linear in the size of the lattice and also a sublinear number of entangling gates.

For an orientable, connected, compact surface of genus g, it is well known that $H_1(\Gamma, \mathbb{F}_d) = (\mathbb{F}_d)^{2g}$. (See e.g. [14].) We next describe how one might transfer a qudit $|\psi\rangle$ stored within an ancilliary copy of $\mathcal{H}(1,d)$ to the topologically ordered ground-state eigenspace of H, say $\mathcal{H}_{gr} \cong \mathbb{C}^{d^{2g}}$.

Since section 4.2 describes how to swap a pure state of $\mathcal{H}_{gr}$ with any ancilla state, the case $|\psi\rangle = |0[\omega]\rangle$ would suffice. We treat generic $|\psi\rangle$ since this is not (much) more difficult than the base case. Also, without a swap, the fact that $|\psi\rangle$ is topologically protected makes the coded state inconvenient to manipulate. Since the topologically ordered groundspace is a stabilizer code, Pauli tensors in the normalizer may be applied without carrying data out of the code. These are not universal, and general operations would be more convenient after an ancilla swap. Hence, a certain amount of effort might be saved by storing a desired $|\psi\rangle$ at the outset.

We begin with $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle$. Choose a copy of $\mathbb{F}_d \subseteq H_1(\Gamma, \mathbb{F}_d)$, and let $[\omega]$ correspond to $1 \in \mathbb{F}_d$. Choose $\omega \in [\omega]$, preferably with as few nonzero (excited) edges as

possible. Now $j\omega$ is also a cycle for $0 \leqslant j \leqslant d-1$, and by our choice $\{[j\omega] = j[\omega]\}_{j=0}^{d-1}$ contains distinct homology classes. Using whatever unitaries are convenient, we form

$$|\tilde{\psi}\rangle = \sum_{j=0}^{d-1} \alpha_j |j\omega\rangle. \tag{12}$$

For example, on a toric Γ one might have n-sites and choose a vertical or horizontal cycle on a square fundamental domain. Then the appropriate unitary would cost $O(\sqrt{n})$ gates. Our goal is to construct $|\psi_{\text{storage}}\rangle = \sum_{j=0}^{d-1} \alpha_j |[j\omega]\rangle$. For the remainder of the construction, note that all intermediate states are in the code space of the stabilizer $G_v = \langle\{g_v\}\rangle \subseteq \mathcal{P}(n,d)$. Hence, we may correct for errors in this code at any time. Also, the scheme below might be thought of as arising from an error correction to the stabilizer $G_f = \langle\{g_f\}\rangle$. Nonetheless, only $|\psi_{\text{storage}}\rangle$ is in the code space of the full stabilizer G. Arbitrary local errors are correctible in the code space of G since the normalizer of G contains $\{Z_e, X_e; e \in \mathcal{E}\}$ [8, 11]. Since this is clearly false for G_v, one should perform the initialization above as quickly as possible.

We suppose an ordering of the faces $f \in \mathcal{F}$, say $f_1, f_2, \ldots, f_L$, such that for each fixed ℓ the boundary of f_ℓ contains some edge e_ℓ which (i) is not within the boundaries of $f_1, f_2, \ldots, f_{\ell-1}$ and (ii) does not intersect the support of ω. This is not possible for the last face f_L, but we only require this condition for $1 \leqslant \ell \leqslant L-1$. To store the qudit beginning with $|\tilde{\psi}\rangle$, we apply the following steps for each f_ℓ.

- Measure the eigenvalue of g_{f_ℓ}, e.g. using an ancillary qudit. (See section 2.2.) The eigenvalue λ will be an element of $\{\xi^j\}_{j=0}^{d-1}$.
- If $\lambda = 1$, then the state has collapsed onto the stabilizer $\langle\{g_v\} \sqcup \{g_{f_k}; 1 \leqslant k \leqslant \ell\}\rangle$ (by induction). Else, measuring ξ^j accidentally performed the collapse $P_j = (1/d)\sum_{k=0}^{d-1} \xi^{jk} g_{f_\ell}^k$, which is in fact a projection[3]. Let e_ℓ be the isolated edge as above. Since $Z_{e_\ell}^k g_{f_\ell} = \xi^k g_{f_\ell} Z_{e_\ell}^k$, we see that $Z_{e_\ell}^j P_j = P_0 Z_{e_\ell}^j$. Thus an appropriate power of Z_{e_ℓ} will fix the projection onto the unwanted eigenvalue so that the final state lies within the $+1$ eigenspace of H_{f_ℓ}.

Applying the process of the last paragraph clearly produces an element of $\mathcal{H}_{gr}$. The applications of H_f, P_j, and also Z_e all respect $\mathcal{H}_{[j\omega]}$ for $0 \leqslant j \leqslant d-1$. Note that $\mathcal{H}_{gr} \cap \mathcal{H}_{[j\omega]} = \mathbb{C}|[j\omega]\rangle$. If S denotes the superoperator of the above sequence of measurements and unitary maps, then equivalently we have shown $S(\mathcal{H}_{\text{loop}}) \subseteq \mathcal{H}_{gr} \cap \mathcal{H}_{[j\omega]}$. Equality is immediate after noting $S|[j\omega]\rangle = |[j\omega]\rangle$.

However, the effect of the superoperator on relative phases is still unclear. Given the global phase on $|\omega\rangle$, there is a natural global phase on $|[\omega]\rangle = \pi|\omega\rangle$. With the argument above, we have actually verified that $S|0\rangle = \mathrm{e}^{\mathrm{i}\varphi_0}|[0]\rangle$, $S|\omega\rangle = \mathrm{e}^{\mathrm{i}\varphi_1}|[\omega]\rangle$, $S|2\omega\rangle = \mathrm{e}^{\mathrm{i}\varphi_2}|[2\omega]\rangle$, etc. Thus perhaps $|\psi_{\text{storage}}\rangle = \sum_{j=0}^{d-1} \mathrm{e}^{\mathrm{i}\varphi_j}\alpha_j|[j\omega]\rangle$. We argue that all of these relative phases are in fact equal. For in terms of the observed eigenvalues,

$$S = \prod_{j=1}^{L-1} Z_{e_\ell}^{\pm j} P_j(f_\ell) = \prod_{j=1}^{L-1} P_0(f_\ell) Z_{e_\ell}^{\pm j} = \pi \prod_{\ell=1}^{L-1} Z_{e_\ell}^{\pm j}. \tag{13}$$

By choice of the support of $|\omega\rangle$, also the support of $|j\omega\rangle$, we have $\prod_{\ell=1}^{L-1} Z_{e_\ell}^{\pm j}|j\omega\rangle = 1$. Thus, applying the superoperator S to $|\tilde{\psi}\rangle$ produces $|\psi_{\text{storage}}\rangle = \sum_{j=0}^{d-1}\alpha_j|[j\omega]\rangle$, given that we may choose the $\{e_\ell\}_{\ell=1}^{L-1}$ to be disjoint from the support of ω.

[3] Why is this a projection? Consider the unitary $h = \xi^j g_{f_\ell}$ and consider projection onto the stabilizer of $\langle\{h\}\rangle$.

4.2. Retrieval

Thus we next consider retrieval of a qudit stored as in the last subsection, i.e. swapping the data in a topological qudit with that encoded in some ancilla qudit. Physically, this is more intricate than encoding, which amounts to creating a cycle class $|\omega\rangle$ and then applying stabilizer corrections for $\{g_f\}_{f\in\mathcal{F}}$ generating G_f.

For retrieval, the central point is that we may apply a logical X operation to the encoded qudit using $O(\sqrt{n})$ gates. To see this, for $\omega = \sum_e n_e e$ let $X^{\otimes\omega} = \otimes_{e\in\mathcal{E}} X_e^{n_e}$. This might be thought of as a creation operator of an excitation of the loop ω, and moreover $X^{\otimes\omega}$ is an element of the centralizer of G not contained within G. As such, it preserves the code space, and one readily verifies that it must map $|[j\omega]\rangle \mapsto |[(j+1)\omega]\rangle$, up to global phase. Hence, we may apply controlled-X operations targeting the topological qudit using $O(\sqrt{n})$ physical controlled-X operations.

We next consider a controlled-X operation controlled on the topological qudit and targeting an ancilla. One must choose a cycle in the dual complex to Γ according to ω, say η. For example, an earlier work [5, figure 3] depicts a picket fence dual to a loop generator of the first homology group of a torus. In order to perform the required controlled-X, follow these steps.

- Prepare a second ancilla. Then prepare this second ancilla so that the Z eigenstate of the ancilla measures $Z^{\otimes\eta}$.
- Perform the controlled-X contingent on this second ancilla.
- Disentangle, i.e. reverse the qudit gates of the first step.

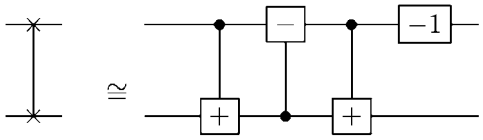
Consequently, we can perform either controlled-X to or from the topologically encoded qudit.

The ability to perform a two-qudit controlled-X gate implies the ability to perform controlled modular addition. The composition begins with a single controlled increment triggering when the control carries $|1\rangle$, continues with two controlled increments when the control carries $|2\rangle$, etc. The entire circuit thus realizes a controlled modular addition in a number of controlled-X gates roughly the triangular number of d. Controlled modular subtraction is similar.

Finally, modular addition and subtraction allow us to SWAP the topological qudit to an ancilla. For bits, the standard three CNOT swap relies on the fact that CNOT exclusive-or's one bit to another. Thus the CNOTs perform $b_1 b_2 \mapsto b_1(b_1 \oplus b_2) \mapsto b_2(b_1 \oplus b_2) \mapsto b_2 b_1$. In like manner, we may perform suitably controlled and targeted additions and subtractions for the following sequence of dit operations:

$$d_1 d_2 \mapsto d_1(d_1 + d_2) \mapsto (-d_2)(d_1 + d_2) \mapsto (-d_2)d_1 \tag{14}$$

Hence, modifying gates so that a control symbol with a + or − target means to add or subtract the control respectively, we have the following diagram:



We have not described how to complete the gate $|j\rangle \mapsto |d-j\rangle$ on the topologically ordered state. Rather than do so, we claim the top line as the ancilla. This also improves the cost of the controlled additions.

4.3. Modified constructions using ancillary qudits

In the quantum circuit model of computation ancillary particles are often used as a means to assist in gate operations and as an entropy dump during error correction cycles. In the

context of surface codes it is tempting to borrow this idea and place qudits at the centre of each face and on each vertex of the cellulation Γ, so that the appropriate stabilizer checks might be done in place (see figure 1(*b*)). Recall, any state may be projected into the ground state of the topologically ordered Hamiltonian $H = H_\partial + H_{\mathrm{KE}}$ using stabilizer checks to the Pauli tensors $\{g_v\} \sqcup \{g_f\} \subset \mathcal{P}(n, d)$ (section 3.2). Each individual stabilizer check may then be performed using a certain sequence of two-qudit gates and a neighbouring ancilla (section 2.2). In fact, this basic observation presents an auxiliary Hamiltonian which also computes the same topological order as the original. Namely, on the face-edge-vertex qudit system, one may build a Hamiltonian which is in the ground state iff all the stabilizer checks g_v and g_f are satisfied. For g_v, suppose we use Σ_e^v for the sum gate targeting the qudit of vertex v and take $\mathbf{n}_v$ to be the qudit number operator on v. Then

$$\tilde{H}_v = \prod_{[*,v]=e} \Sigma_e^v \prod_{[v,*]=e} \left(\Sigma_e^v\right)^{-1} \mathbf{n}_v \prod_{[*,v]=e} \left(\Sigma_e^v\right)^{-1} \prod_{[v,*]=e} \Sigma_e^v. \tag{15}$$

Then $|\psi\rangle$ is in the ground state of $\tilde{H}_v$ iff $g_v|\psi\rangle = |\psi\rangle$. Similarly, fix a face $f \in \mathcal{F}$ with $\partial f = \sum_{j=1}^{\ell} n_j e_j$ for $n_j \in \{1, d-1\}$. We take $\mathbf{F_f} = \prod_{j=1}^{\ell} (\mathcal{F}_d)_{e_j}$ and $U_f = \prod_{j=1}^{\ell} \left(\Sigma_{e_j}^f\right)^{n_j}$ for Σ_e^f the sum gate targeting the f qudit. Then for $\mathbf{n}_f$ the number operator of the qudit at the center of the face f, we label

$$\tilde{H}_f = \mathbf{F_f} U_f \mathbf{F_f}^\dagger \mathbf{n}_f \mathbf{F_f} U_f^\dagger \mathbf{F_f}^\dagger. \tag{16}$$

As before, we see that $|\psi\rangle$ is in the ground state of $\tilde{H}_f$ iff $g_f|\psi\rangle = |\psi\rangle$. Thus for $h > 0$ and $U > 0$, if $\tilde{H} = U \sum_v \tilde{H}_v + h \sum_f \tilde{H}_f$, then the ground state of $\tilde{H}$ is also the code space of $G = \langle \{g_v\} \sqcup \{g_f\} \rangle$, i.e. the topologically ordered ground state spanned by $\{|[\omega]\rangle; [\omega] \in H_1(\Gamma, \mathbb{F}_d)\}$.

We finish this section describing another utility for ancillary particles, that is to mediate many body interactions present in the Hamiltonian H (equation (6)) using more physically motivated binary interactions. Consider the vertex constraint term $H_v = -\left(g_v + g_v^\dagger\right)$ where the valence at that vertex is k. This k-local interaction can be obtained as a perturbative limit of 2-local interactions between each d-level qudit incident at v and a k-level ancillary qudit a located at the vertex. Begin with a local Hamiltonian for the ancilla $H_a = -E_a|0\rangle_a\langle 0|$, and a perturbing interaction $V_a = J_v \sum_{r=1}^{k} \left(Z_{e_r}^{o_r} \otimes |r-1\rangle\langle r| + \text{h.c.}\right)$, where $E_a \gg |J_v|$ and the edge orientations give $o_j = 1$ if $e_j = [*, v]$ and $o_j = d-1$ if $e_j = [v, *]$. By construction, the lowest nontrivial, i.e. non-identity, contribution to coupling in the ground subspace spanned by all ancilla in state $|0\rangle$ is the effective Hamiltonian $H_{v\,\mathrm{eff}} = U(H_v + O(\epsilon))$ where $U = (-1)^k E_a (J_v/E_a)^k$ with an error term of norm $\|\epsilon\| \ll 1$. By judicious choice of sign(J_v) it is possible to fix $U > 0$. A similar argument applies to building the face constraint H_f using a j-level ancilla b located at face f to mediate interactions between all j edges on the boundary of f. Here we choose $H_b = -E_b|0\rangle_b\langle 0|$ and $V_b = J_f \sum_{r=1}^{j} \left(X_{e_r}^{o_r} \otimes |r-1\rangle\langle r| + \text{h.c.}\right)$ such that $H_{f\,\mathrm{eff}} = h(H_f + O(\epsilon))$, where $h = (-1)^j E_b (J_f/E_b)^j$. These mediator qudits could be placed on all the vertexes and faces of Γ to build an effective Hamiltonian in the subspace spanned by states with all ancillae in the $|0\rangle$ state.

It was proven in [10, 17] that any k-local Hamiltonian for constant k acting on spins whose coupling graph is a two-complex, can be closely approximated by the low energy projection of a Hamiltonian built using only 2-local interactions with mediator particles. Here close means that $\|H_{\mathrm{eff}} - H_{0\,\mathrm{target}}\| \ll \Delta E$, where H_{eff} is the 2-local Hamiltonian H projected into the ground states of the mediator particles, $H_{0\,\mathrm{target}}$ is the target k-local Hamiltonian restricted to its low energy subspace, ΔE is the energy gap between the ground and excited states of $H_{0\,\mathrm{target}}$, and $\|\cdot\|$ is a suitable operator norm. In the context of building the surface code Hamiltonian this would imply that for sufficiently large energies E_a, E_b that scale linearly with the system

size n, both the degeneracy of the ground subspace of $H_{\text{target}} = H_\partial + H_{\text{KE}}$ and the energy gap to the excited states could be accurately approximated by $H_{\text{eff}} = \sum_v H_{v\,\text{eff}} + \sum_f H_{f\,\text{eff}}$. An analysis regarding the accuracy of such constructions for building topologically ordered states is wanting, but is beyond the scope of this work.

5. Other homological ground states

We have originally presented the case of ground states for $H_1(\Gamma, \mathbb{F}_d)$ for d prime, in order to present the new orientation conventions in the simplest possible context. This section describes a construction for homological order on dits whose number of levels is not prime but rather a prime power. Homological order for arbitrary composite d follows immediately through a tensor product of the prime-power Hamiltonians.

5.1. Homology $\mathbb{F}_{d^\ell}$ stabilizer codes

The hypothesis in the main text has been that qudits have d levels, for d a prime so that each $|j\rangle$ is associated to an element of $\mathbb{F}_d$. Recent work [11] extends stabilizer techniques to the finite fields of order d^ℓ, i.e. $\mathbb{F}_{d^\ell}$, which exist for any $\ell \geqslant 1$.

The generic $\mathbb{F}_{d^\ell}$ constitute all fields $\mathbb{F}$ with $\#\mathbb{F} < \infty$, so this is (perhaps) the most general field for which a stabilizer code makes sense. The most typical construction of $\mathbb{F}_{d^\ell}$ is to consider the polynomial ring $\mathbb{F}_d[x]$ and divide out relations in the ideal generated by some irreducible polynomial $f(x) = x^\ell + a_{\ell-1}x^{\ell-1} + \cdots + a_0, a_j \in \mathbb{F}_d$. It is typical to label $\alpha \in \mathbb{F}_{d^\ell}$ as the adjoined root corresponding to the class of x. The Galois group of the extension $\mathbb{F}_{d^\ell}$ over $\mathbb{F}_d$, say K, then acts as permutations of the roots of $f(x)$. Note that $\mathbb{F}_{d^\ell}$ is a vector space over the scalars $\mathbb{F}_d$. Moreover, multiplication by any fixed $a \in \mathbb{F}_{d^\ell}$ may be viewed as a $\mathbb{F}_d$-linear map, with an associated matrix with entries in $\mathbb{F}_d$. Computing the trace of this matrix creates a map $\{\text{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d} : \mathbb{F}_{d^\ell} \to \mathbb{F}_d\}$. Another characterization is that $\text{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d}(x) = \sum_{\kappa\in K}(\kappa \cdot x)$. To ground the discussion, let us review not extensions over finite fields but rather $\text{Trace}_{\mathbb{C}/\mathbb{R}}(z) = z + \overline{z} = 2\text{Re}(z)$. The complex conjugate is the Galois action that interchanges $i \leftrightarrow -i$, for $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$. We might instead form a 2×2 matrix for multiplication by $z = x + \text{i}y$, which results in $\mu_z = x|0\rangle\langle 0| - y|1\rangle\langle 0| - y|0\rangle\langle 1| + x|1\rangle\langle 1|$ with trace $2x$.

For $\mathbb{F}_{d^\ell}$ extending $\mathbb{F}_d$ the Galois group K is cyclic of order ℓ, generated by $x \overset{\kappa}{\mapsto} x^d$ for $x \in \mathbb{F}_{d^\ell}$. Now κ generates a one-qud$^\ell$ it unitary U_κ by $U_\kappa|x\rangle = |\kappa \cdot x\rangle$, and the corresponding diagonal unitary on the entire lattice will be denoted by $\tilde{U}_\kappa$.

5.1.1. Fourier transforms for $\mathbb{F}_{d^\ell}$. Having reviewed the machinery of finite fields, we next review what one would mean by a stabilizer code of Pauli matrices indexed by it [11]. Since our earlier qudit operators X and Z for $\mathbb{F}_d$ had order d, we might instead claim to have constructed an X operator and a Z operator for each $a \in \mathbb{F}_d$, i.e. X^a and Z^b. For $\mathbb{F}_{d^\ell}$, we do not take operator powers. Label $\mathcal{H}(1, d^\ell) = \oplus_{a\in\mathbb{F}_{d^\ell}}\mathbb{C}\{|a\rangle\}$. Then suitable definitions are as follows, where we define $\xi = \exp(2\pi\,\text{i}/d)$:

$$\begin{cases} X(a)|b\rangle = |a+b\rangle \\ Z(a)|b\rangle = \xi^{\text{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d}(ab)}|b\rangle. \end{cases} \tag{17}$$

For $\ell = 1$, this generalizes the powers of earlier Pauli operators. Furthermore, with these conventions we have a commutator relation

$$X(a)Z(b) = \xi^{\text{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d}(ab)}Z(b)X(a). \tag{18}$$

Finally, let $\mathcal{H}(n, d^\ell) = \mathcal{H}(1, d^\ell)^{\otimes n}$. In a slight abuse of notation, for $a, b \in (\mathbb{F}_{d^\ell})^n$ we will write $a \bullet b = \mathrm{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d}(a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1})$. Then we may generalize the earlier commutator formula for Pauli tensors as

$$\begin{aligned}&[Z(b_0) \otimes Z(b_1) \otimes \cdots \otimes Z(b_{n-1})] \\ &[X(a_0) \otimes X(a_1) \otimes \cdots \otimes X(a_{n-1})] = \xi^{a \bullet b} [X(a_0) \otimes X(a_1) \otimes \cdots \otimes X(a_{n-1})] \\ &\qquad \times [Z(b_0) \otimes Z(b_1) \otimes \cdots \otimes Z(b_{n-1})]. \end{aligned} \tag{19}$$

Given this relation, one may define $\mathcal{P}(n, d^\ell)$ to be that group generated by products of Pauli tensors indexed by $\mathbb{F}_{d^\ell}$, as above. Continuing, we may consider sets of particular Pauli tensors $X(a_0)Z(b_0) \otimes \cdots \otimes X(a_{n-1})Z(b_{n-1})$ and consider the stabilizer subspaces of the subgroup $G \subset \mathcal{P}(n, d^\ell)$ they generate. The error lengths of such code are studied in detail [11].

Before considering which of these stabilizer codes arise as topological orders, we add a point omitted in the original treatments. Namely, we wish to propose quantum circuits for the appropriate stabilizer checks. We suppose the existence of a number operator measurement which can output classical values in the finite field, say abusively $n = \sum_{a \in \mathbb{F}_{d^\ell}} a|a\rangle\langle a|$. Then as with $\mathbb{F}_d$, stabilizer checks would follow given an appropriate Fourier transform $\mathcal{F}_{d^\ell} : \mathcal{H}(n, d^\ell) \to \mathcal{H}(n, d^\ell)$ which maps $X(a)$ eigenstates to $|a\rangle$. This leads one to guess we should define $\mathcal{F}_{d^\ell}|a\rangle = (d^\ell)^{-1/2} \sum_{b \in \mathbb{F}_{d^\ell}} Z(a)|b\rangle$, i.e.

$$\mathcal{F}_{d^\ell} \stackrel{\mathrm{def}}{=} (d^\ell)^{-1/2} \sum_{a,b \in \mathbb{F}_{d^\ell}} \xi^{\mathrm{Trace}_{\mathbb{F}_{d^\ell}/\mathbb{F}_d}(ab)} |b\rangle\langle a|. \tag{20}$$

However, note that $X(a)$ now has degenerate eigenspaces when $\ell \geqslant 2$. Thus, it is not clear whether the above equation actually defines a unitary matrix.

We briefly comment on why unitarity holds. For convenience, let us drop the subscript from the appropriate trace maps. A computation reveals that the unitarity assertion is equivalent to knowing that for any fixed $a \in \mathbb{F}_{d^\ell}$ which is nonzero,

$$\sum_{b \in \mathbb{F}_{d^\ell}} \xi^{\mathrm{Trace}(ab)} \stackrel{?}{=} 0. \tag{21}$$

Since $a \neq 0$ has a multiplicative inverse, this amounts to

$$\sum_{b \in \mathbb{F}_{d^\ell}} \xi^{\mathrm{Trace}(b)} \stackrel{?}{=} 0. \tag{22}$$

Now suppose we use α to denote the formally adjoined root of $f(x)$ in $\mathbb{F}_{d^\ell} = \mathbb{F}_d[x]/(f(x))$. Then since every equivalence class may be written as a polynomial of degree less than ℓ, we see that $\{\alpha^j\}_{j=0}^{\ell-1}$ is a basis of $\mathbb{F}_{d^\ell}$ over $\mathbb{F}_d$. In terms of the last basis, we might express a generic polynomial class in coordinates as $b = b_{\ell-1}x^{\ell-1} + b_{\ell-2}x^{\ell-2} + \cdots + b_0$ for $b_j \in \mathbb{F}_d$. Then equation (22) becomes

$$\sum_{b_{\ell-1}=0}^{d-1} \sum_{b_{\ell-2}=0}^{d-1} \cdots \sum_{b_0=0}^{d-1} [\xi^{\mathrm{Trace}(\alpha^{\ell-1})}]^{b_{\ell-1}} [\xi^{\mathrm{Trace}(\alpha^{\ell-2})}]^{b_{\ell-1}} \cdots [\xi^{\mathrm{Trace}(1)}]^{b_0} \stackrel{?}{=} 0. \tag{23}$$

This will in fact be zero, unless all $\mathrm{Trace}(\alpha^j) = 0 \bmod p, 0 \leqslant j \leqslant \ell - 1$. A standard construction in field extensions is to form the discriminant of a basis, for our basis $\Delta = \sum_{j,k=0}^{\ell-1} \mathrm{Trace}(\alpha^{j+k})|k\rangle\langle j|$. For a given basis, it is not possible that this matrix Δ has determinant zero in $\mathbb{F}_d$ [theorem 2.37, p 61][16]. Since the first column of Δ cannot then be zero, all $\mathrm{Trace}(\alpha^j)$ may not be zero, and unitarity of $\mathcal{F}_{d^\ell}$ follows.

5.1.2. Homological order for $\mathbb{F}_{d^\ell}$. The chain complex for computing $H_1(\Gamma, \mathbb{F}_{d^\ell})$ extends our early discussion by allowing for coefficients of the vertices, edges and face to be within $\mathbb{F}_{d^\ell}$, which in the context is ℓ copies of $\mathbb{F}_d$ since only the additive structure is relevant. Yet the previous section has nontrivially extended our definition of X and Z operators to account for field multiplication, and these operators may be used to form a homological order on the physical system in which qud^ℓ its (with d^ℓ levels) are associated to the edges of Γ:

- For each vertex, we may again set $g_v = \prod_{e=[*,v]} Z_e(1) \prod_{e=[v,*]} Z_e(-1)$ and $H_v = -\left(g_v + g_v^\dagger\right)$. Then again $H_\partial = U \sum_{v\in\mathcal{V}} H_v$.
- Again set $g_f = X_{e_1}(o_1)X_{e_2}(o_2)X_{e_3}(o_3)\cdots X_{e_p}(o_p)$, where $\partial f = \sum_{\ell=1}^{p} o_\ell e_\ell$. Put $H_f = -\left(g_f + g_f^\dagger\right)$. Given the generalization of the commutators of the new X and Z operators, $[H_f, H_v] = 0$ for any f, v. Then for $h > 0$, $H_{\mathrm{KE}} = h \sum_{f\in\mathcal{F}} H_f$.
- So $H = H_\partial + H_{\mathrm{KE}}$. A similar argument to that given before produces a basis $|[\omega]\rangle$ of the groundspace of H, as $[\omega]$ runs over all elements of $H_1(\Gamma, \mathbb{F}_{d^\ell})$.
- These ground states may again be viewed as a stabilizer code of $G = \langle\{g_v, g_f\}\rangle \subsetneq \mathcal{P}(n, d^\ell)$. Stabilizer checks can be performed as before (see section 2.2). The only required modifications are that the quantum circuit uses the new Fourier transform over $\mathbb{F}_{d^\ell}$ to measure $X(a)$ operators and powers thereof and the number operator measurement now takes values in $\mathbb{F}_{d^\ell}$.

We close with one further comment. Recall $\tilde{U}_\kappa$ which act on each qud$^\ell$ it as $U_\kappa|a\rangle = |\kappa\cdot a\rangle$ for κ the generator of the cyclic Galois group of $\mathbb{F}_{d^\ell}$ extending $\mathbb{F}_d$. Now $\tilde{U}_\kappa H = H\tilde{U}_\kappa$, as one can verify directly using H_f and H_g. Thus we may view $\tilde{U}_\kappa$ or more generally the Galois action as a symmetry of the topologically ordered ground state. Also, $\pi_\kappa = \ell^{-1}\sum_{j=0}^{\ell-1} U_\kappa^j$ will then act as a projection collapsing the ground state associated to elements of $H_1(\Gamma, \mathbb{F}_{d^\ell})$ onto the ground state parametrized by $H_1(\Gamma, \mathbb{F}_d)$ as constructed in section 3. In terms of Hamiltonians, π_κ projects onto the ground state of $H_\kappa = -\left(U_\kappa + U_\kappa^\dagger\right)$, whose physical significance is unclear.

6. $\mathbb{Z}/d\mathbb{Z}$ gauge theory and anyonic excitations

In our treatment of code subspaces, we have used the isomorphism between spins on a surface and one-chains on a two-complex to label the ground states of the Hamiltonian H in terms of homology equivalence classes. The language of cell complexes also carries over to describe the excited states. If we identify the ground subspace of H as the vacuum then excited states are labelled by $\mathbb{F}_d$ valued boundaries of one chains on the complex Γ or the dual complex $\tilde{\Gamma}$. These excitations can be viewed as massive particles with definite statistics.

In this section we show by construction that our model is a $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge theory. Excitations correspond to quasi-particles which are described by dyonic combinations of charge and flux with Abelian anyonic statistics. We provide an algorithm in terms of an interferometer circuit for measuring components of the scattering matrix for these particles.

6.1. Stabilizer errors as Abelian anyons

Consider a two-complex Γ with a physical system of qudits associated to each edge and a topologically ordered Hamiltonian H as above. We have already seen how to associate a basis of the ground-state eigenspace with elements of $H_1(\Gamma, \mathbb{F}_d)$. As stabilizer states, it is well known that the ground states are entangled. Abelian anyons arise as entangled excitations of this system. In the qubit case, such excitations always arise in pairs [12]. In our generalization, this is also true, and an excitation $|j\rangle$ is always paired to an excitation $|d - j\rangle$.

The linear algebra for constructing a charge anyon is as follows. First, choose two vertices v_1 and v_2 of Γ on which the anyon should reside with charges j and $d-j$ respectively. Choose a chain ω with $\partial\omega = jv_1 + (d-j)v_2$. Recall from section 3.2 the projection $\pi = (\#\mathcal{F})^{-1}\sum_{f\in\mathcal{F}} g_f$ which projects onto the stabilizer code of all the face operators $g_f = \otimes_{e\in\partial f}(X_e)^{\pm}$. We set

$$|\psi_{\text{charge anyon}}\rangle = \pi|\omega\rangle. \tag{24}$$

The resulting state is an excited state of H_∂ whose eigenenergy is $4U(1-\mathrm{Re}(e^{2i\pi j/d}))$ above ground. It is *not* independent of the choice of ω, and this in fact allows for an interesting geometric interpretation of the error length of the associated stabilizer code [5].

Let ω_1 and ω_2 be two such choices, with $|\psi_1\rangle$ and $|\psi_2\rangle$ the resulting anyon states. Then $\omega_1-\omega_2$ is a cycle, and

$$|\phi\rangle \stackrel{\text{def}}{=} \pi(|\omega_1\rangle - |\omega_2\rangle) = |\psi_1\rangle - |\psi_2\rangle \tag{25}$$

is the ground-state eigenket associated to $[\omega_1-\omega_2]\in H_1(\Gamma,\mathbb{F}_d)$. Hence, if we encounter such a charge anyon excitation which has sullied a qudit encoded in the ground state of H, then correcting it amounts to choosing an cancelling anyon or equivalently to choosing a cycle on Γ. If the dual charges of the anyon are separated by roughly half the diameter of the two-complex, then this choice is likely to cause an error. Yet for nearby dual charges one might reasonably guess $[\omega_1-\omega_2]=[0]$. In particular, if Γ were to cellulate the square fundamental domain of a torus using n qudits on the edges (implying $\Theta(\sqrt{n})$ qudits on a side), then we would expect an error length for the associated stabilizer code to be roughly $O(\sqrt{n})$ [5].

Similar comments apply not only to charge anyons but also flux anyons [12]. Here, one chooses a path in the dual complex to Γ, i.e. a sequence of connected faces. Let $|[0]\rangle$ be the homological ground state associated to $[0]\in H_1(\Gamma,\mathbb{F}_d)$. A flux charge of multiplicity j on the endpoints of the face path is associated to

$$|\psi_{\text{flux anyon}}\rangle = \pi_v g_{f_1}^{\pm j} g_{f_2}^{\pm j}\cdots g_{f_\ell}^{\pm j}|[0]\rangle \tag{26}$$

where $\pi_v = (\#\mathcal{V})^{-1}\sum_{v\in\mathcal{V}} g_v$ and the path consists of faces $f_1, f_2, \ldots, f_\ell$ with the signs allowing for orientation. The flux anyon theory follows quickly by considering the charge anyons of the dual two-complex to Γ, say $\tilde{\Gamma}$. Faces of Γ become vertices of $\tilde{\Gamma}$ while vertices become faces, and the graph of $\tilde{\Gamma}$ arises by connecting vertices corresponding to incident faces of Γ. Suitable hypotheses on the cellulation of the underlying two-manifold of Γ will cause this dualization procedure to be well behaved [14], and one might associate charge-anyonic observation of flux anyons and vice versa with pairings exploited in the proof of Poincaré duality.

6.2. *Quasi-particle statistics*

We next wish to study such anyon states, i.e. errors of the stabilizer code as above. New notation for the excitations follows. A charge $a\in\mathbb{Z}/d\mathbb{Z}$ at vertex v is labeled by the state $|(a,0;(v,-))\rangle$ such that $\langle(a,0;(v,-))|g_v|(a,0;(v,-))\rangle = \xi^a$. Similarly, flux $b\in\mathbb{Z}/d\mathbb{Z}$ at face f is labeled by the state $|(0,b;(-,f))\rangle$ such that $\langle(0,b;(-,f))|g_f|(0,b;(-,f))\rangle = \xi^b$. A dyon refers to a bound state of charge and flux at vertex v and face f neighboring each other, i.e. $[v,*]\in\partial f$ or $[*,v]\in\partial f$ and $(a,b)\in(\mathbb{Z}/d\mathbb{Z})^2$. The state of such a dyon in Hilbert space will be denoted by $|(a,b;(v,f))\rangle$. For simplicity we restrict our discussion to simply connected compact surfaces with boundary such that the ground (vacuum) state is nondegenerate[4].

[4] In general $|(a,b);(v,f)\rangle$ describes an equivalence class of pure states which results from applying $X_e^a Z_e^{-b}$ to *any* ground state. For a degenerate vacuum, particle creation, followed by braiding and annihilation can result in non-trivial logical operations on the code subspace.

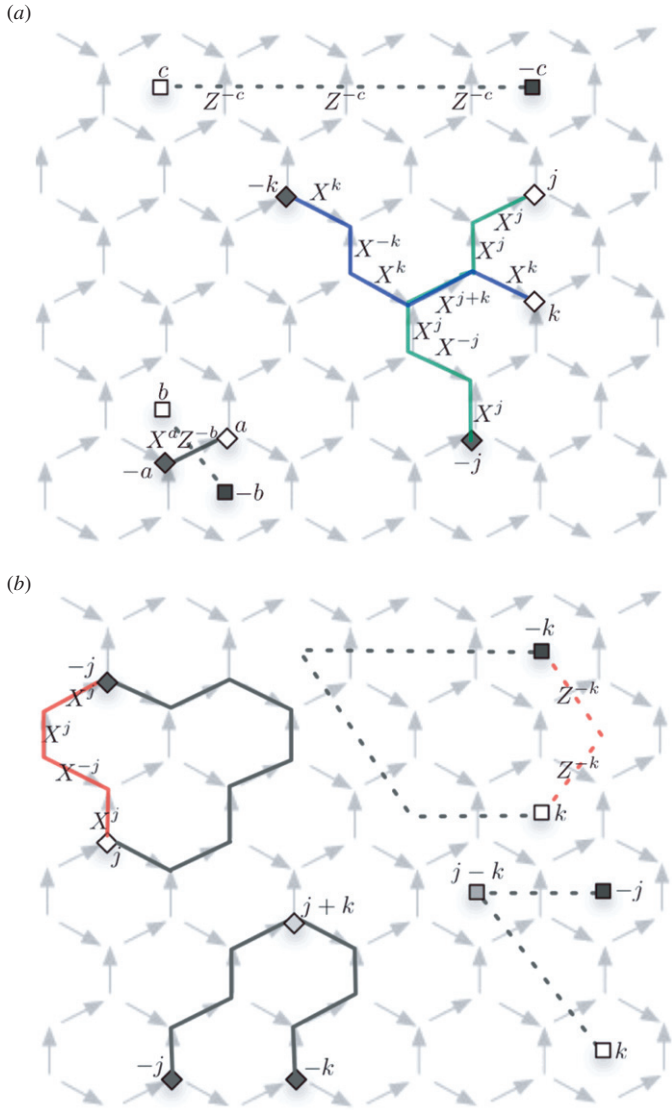


Figure 3. Quasi-particle excitations on a honeycomb cellulation. (*a*) Excitations appear in particle–antiparticle pairs. Charges(anti-charges) appear as boundaries on vertices represented by open(filled) diamonds, and fluxes(anti-fluxes) as boundaries on the faces represented by open(filled) squares. The total charge and flux of any pair is zero. Shown is a flux pair $|(0, c); (0, -c)\rangle$, charge pairs $|(j, 0); (j, 0)\rangle$, $|(k, 0); (-k, 0)\rangle$ and a bound state of charge and flux pairs $|(a, b); (-a, -b)\rangle$. Note that strings of the same or different types are allowed to intersect. (*b*) Fusion of quasi-particles. The upper two diagrams illustrate corrective procedures to annihilate charge and flux excitations. The lower two diagrams illustrate the fusion rules $|(j, 0)\rangle \times |(k, 0)\rangle = |(j+k, 0)\rangle$ and $|(0, j)\rangle \times |(0, -k)\rangle = |(0, j-k)\rangle$.

Pauli-group elements local to a single edge of Γ produce dyons of the topological order in particle–antiparticle pairs. To see this, note that the operator X_e^a acting at edge $e = [v_1, v_2]$ creates a pair of boundaries on the vertices, one with charge a at v_1 and other with charge $d-a$ at v_2. We name the charge $d-a$ particle an anti-charge to a. Similarly, the operator Z_e^b creates

quasi-particles located on the two faces f_1 and f_2 that share the edge e on their boundaries. Let face f_1 be the face with opposite orientation to e. Then the flux at f_1 is b and the anti-flux at f_2 has the value $d-b$. A product operator $X^a Z^{-b}$ acting on edge e creates the dyon (a, b) with charge a at vertex v_1 and flux b at face f_1 (see figure 3(*a*)). When it might be clear from the context, we will drop the particle location labels (v, f), e.g. particle–antiparticle pairs might be written as $|(a, b); (-a, -b)\rangle$. The mass of a dyon is given by the expectation value: $m_{a,b} = \langle (a, b)|H|(a, b)\rangle - E_0 = 2U(1 - \mathrm{Re}[\xi^a]) + 2h(1 - \mathrm{Re}[\xi^b])$, where E_0 is the vacuum energy. The energy to create a particle–antiparticle pair is twice this value.

Prior work in the continuum field theory has considered dyon excitations in which charges and fluxes take values in $\mathbb{Z}/\mathrm{d}\mathbb{Z}$. The interactions described by a $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge theory are completely characterized by the following rules [18]:

$$|(a, b; (v, f))\rangle \times |(a', b'; (v, f))\rangle = |(a + a', b + b'; (v, f))\rangle \tag{27}$$

$$\mathcal{R}|(a, b; (v, f))\rangle|(a, b; (v', f'))\rangle = \xi^{ab}|(a, b; (v, f))\rangle|(a, b; (v', f'))\rangle \tag{28}$$

$$\mathcal{R}^2|(a, b; (v, f))\rangle|(a', b'; (v', f'))\rangle = \xi^{(a'b+b'a)}|(a, b; (v, f))\rangle|(a', b'; (v', f'))\rangle \tag{29}$$

$$\mathcal{C}|(a, b; (v, f))\rangle = |(-a, -b; (v, f))\rangle \tag{30}$$

$$T|(a, b; (v, f))\rangle = \xi^{ab}|(a, b; (v, f))\rangle. \tag{31}$$

We next review these rules and argue that the dyonic excitations of our Hamiltonian satisfy them.

In our construction, local $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge transformations are generated by the unitary vertex and face operators g_v and g_f. The first relation, equation (27), is the fusion rule for particles occupying the same location where addition is performed modulo d. This rule follows from the additivity of boundaries of one chains. Indeed, it is the ability to annihilate particle–antiparticle pairs by choosing a trivial cycle on Γ or $\tilde{\Gamma}$ that makes correction of local errors possible (see figure 3(*b*)). Such cycles are Wilson loops that are intrinsically gauge invariant operators. The next two rules describe the action of the braid operator $\mathcal{R}$ which performs a counterclockwise exchange, or half-braid, of one particle with another. The quantum state of n indistinguishable particles residing on a surface belongs to a Hilbert space that transforms as a unitary representation of the braid group B_n. If we order the positions of the particles $\{(v_j, f_j)\}_{j=1}^n$, then the $n-1$ generators of B_n correspond to the braid operator $\mathcal{R}$ acting on the particle pairs in the locations $\{(v_j, f_j), (v_{j+1}, f_{j+1})\}_{j=1}^{n-1}$. For a $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ gauge theory, the irreducible unitary representation of B_n is one dimensional, meaning the particles are Abelian anyons. Note that the definition of the braid operator involves orientation of the path taken during particle exchange. For a non-orientable surface, $\mathbb{Z}/\mathrm{d}\mathbb{Z}$ statistics for $d > 2$ are not allowed because the clockwise trajectory of particle around another is not uniquely defined whereas the phases ξ, ξ^{-1} are distinguishable except for $d = 2$.

The braiding of one dyon around another is shown in figure 4. Here we begin with a state of two dyonic particle–antiparticle pairs: $|\Psi\rangle = |(a, b); (-a, -b)\rangle|(a', b'); (-a', -b')\rangle$ in distinct locations on the surface. The mutual statistics are determined by winding one dyon, (a, b) around the other (a', b') in a counterclockwise sense. This action is described by the square of the braid operator, or the monodromy operator, which exchanges two particles in a counterclockwise sense. A non trivial phase is accumulated under the action of $\mathcal{R}^2$ because the closed loop string operators that wind (a, b) collide with the strings connected the dyon (a', b') with its antiparticle. In the example shown in figure 4(*a*), the strings intersect at two locations where we have the operators $Z^{-b}X^{-a'} = \xi^{a'b}X^{-a'}Z^b$ and $X^{-a}Z^{b'} = \xi^{b'a}Z^{-b'}X^{-a}$. Rewriting these operators with the action of the closed strings (Pauli operators with unprimed powers)

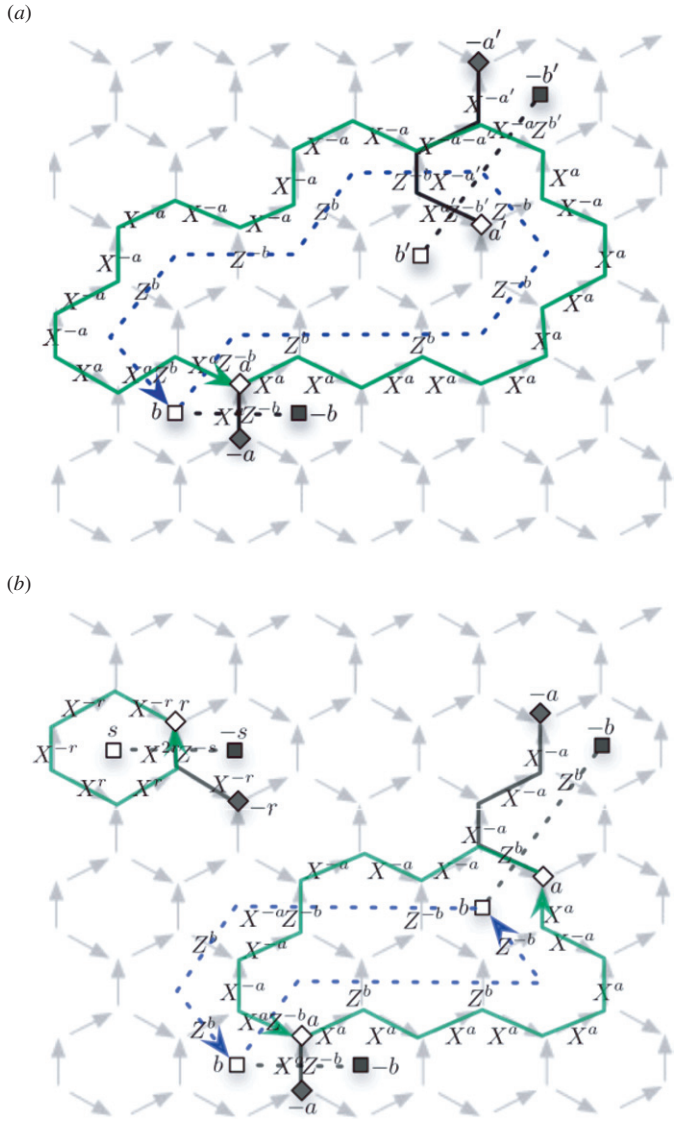


Figure 4. Braid relations. (*a*) Counterclockwise braiding of the dyon (a, b) around the dyon (a', b'): $\mathcal{R}^2|(a,b)\rangle|(a',b')\rangle = \xi^{(a'b+b'a)}|(a,b)\rangle|(a',b')\rangle$. (*b*) Counterclockwise exchange of identical dyons: $\mathcal{R}|(a,b);(a,b)\rangle = \xi^{ab}|(a,b);(a,b)\rangle$. The upper left-hand side of the surface shows the counterclockwise winding of the charge component of a dyon (r, s) about its flux component generating an Aharanov–Bohm phase according to $T|(r,s)\rangle = \xi^{rs}|(r,s)\rangle$.

first has the advantage that the closed strings act trivially provided that there are no other quasi-particles inside the closed loops. Hence we have that $\mathcal{R}^2|\Psi\rangle = \xi^{(a'b+b'a)}|\Psi\rangle$. The preceding example illustrated the Aharanov–Bohm phase accumulated when winding one charge around a flux along a trajectory that was local, i.e. did not explore the global properties of the surface. Were the flux absent, then the path would be homotopic to a point. One can also define this phase for trajectories that explore the global properties of the surface, but can be continuously deformed to a process where one anyon wraps around another. On a torus, for example, the

following process traces out non trivial cycles for the charges and fluxes. Represent the torus as a square with opposite sides identified, labelling the axes of the square x_1, x_2. Pick a non trivial cycle along the x_1 direction of Γ, and call it P_1. Similarly, pick a non-trivial cycle along the x_2 direction of the dual $\tilde{\Gamma}$ and call it P_2. To obtain the exchange statistics, first create the dyonic particle antiparticle pair $|\Psi\rangle = |(a, b; (v_1, f_1)); (-a, -b; (v_2, f_2))\rangle$ out of the vacuum state $|\Psi_g\rangle$. Wind the charge a along P_1 so that it annihilates with its anticharge partner at site v_2. Next wind the flux b along P_2 so that it annihilates with its antiflux partner at face f_2. Create another dyonic particle antiparticle pair $|\Psi'\rangle = |(a, b; (v_2, f_2)); (-a, -b; (v_1, f_1))\rangle$ with particle antiparticle positions reversed relative to $|\Psi\rangle$. Wind charge a along P_1 in the opposite direction to the first winding so that it annihilates with the anticharge at site v_1 and likewise, flux b along P_2 in the opposite direction so that it annihilates with the antiflux at face f_2. These four trajectories cross at one edge e and the action on the state (for one choice of edge orientation) is $|\Psi_g\rangle \to Z_e^{-b} X_e^{-a} Z_e^b X_e^a |\Psi_g\rangle = \xi^{ab} |\Psi_g\rangle$. If we embed the torus in $\mathbb{R}^3$, then the worldlines described by intersecting strings in the above process are equivalent under ambient isotopy to linked worldlines on the plane which describe winding the charge a around the flux b.

Identical quasi-particle statistics are determined by exchanging one dyon (a, b) counterclockwise with another. Such a process is depicted in figure 4(*b*). The action on the state $|\Psi\rangle = |(a, b)\rangle|(a, b)\rangle$ can be computed by annihilating particle–antiparticle pairs after exchange, creating them again and comparing the resultant state with the initial state $|\Psi\rangle$. We can annihilate the charges on the left-hand side first. Reversing the order of the operator that created the dyon there, we have $X^a Z^{-b} = \xi^{ab} Z^b X^a$ and the charges are annihilated by applying Z^b. Similarly, the charges on the right-hand side are annihilated by applying a string of Z^{-b} operators. Finally, the fluxes are annihilated by applying X^a or X^{-a} along the remaining two connected strings. The action on the wavefunction is then $\mathcal{R}|(a, b)\rangle|(a, b)\rangle = \xi^{ab}|(a, b)\rangle|(a, b)\rangle$.

The particle conjugation operator $\mathcal{C}$ in equation (30) reverses the sign of all the particles. This is realized in our microscropic spin model by reversing the orientation of all the edges on the cellulation. Finally, the twist operation T in rule 31 rotates the charge component of a dyon around its own flux, generating an Aharanov–Bohm phase in the process. This is illustrated in figure 4(*b*). Here the charge component of the dyon (r, s) is wrapped around its flux component in a counterclockwise sense. During this operation, there is a collision at the edge where the dyon was created. Rewriting the operation on the edge as $X^{2r} Z^{-s} = \xi^{rs} X^r Z^{-s} X^r$ so that loop operation about boundary of the face f acts trivially first, we have that $T|r, s(v, f)\rangle = \xi^{rs}|r, s(v, f)\rangle$.

6.3. Measuring statistical phases

In any physical construction of a Hamiltonian that admits topologically ordered states it will be important to verify the predicted properties. One, albeit crude, observable is to measure the energy gap from a ground state to a first excited state. This could be done by probing the linear response of the ground states to a perturbing field oscillating at frequency ω_F that generates local unitary operation. For a system with the internal Hamiltonian equation (6), the expected resonant absorption occurs at frequencies $\omega_F = 2m_{a,b}/\hbar$. However, as a witness to topological order, this measure is not sufficient because there could be another spin Hamiltonian with equal gap that does not possess topologically invariant correlation functions. Another more convincing probe would be to directly compute the statistical phases in equation (29). Operationally, this should be done by measuring both the phase ϕ_τ accumulated when one particle (a, b) wraps around another (r, s) and the phase $\phi_{\mathbf{1}}$ when

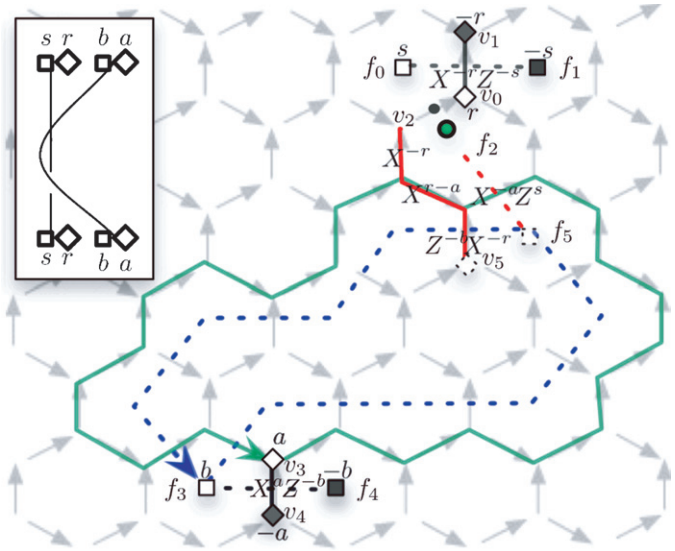


Figure 5. Protocol for measuring quasi-particle statistics. The green circle represents an ancillary particle which performs conditional gate operations on the qudit residing on edge $e = [v_2, v_0]$. The red lines indicate operations which are done adiabatically with respect to the energy gap ΔE. The inset is a simplified space–time diagram of the braid.

the particle (a, b) traces out the same path in configuration space but does not enclose the particle (r, s). The phase difference $\phi_\tau - \phi_{\mathbf{1}} = \phi_{\text{top}}$ subtracts out dynamical phases and Berry's phases, leaving only topological information. We sketch an algorithm for computing this phase using operations in accordance with the two complex illustrated in figure 5. Adaptation to other cellulations is straightforward.

(i) Beginning from a ground state $|\Psi(0)\rangle$, prepare a state with two particle–antiparticle pairs in disjoint regions of the surface:

$$|\Psi(1)\rangle = |(a, b; (v_3, f_3)); (-a, -b; (v_4, f_4))\rangle \\ |(r, s; (v_0, f_0)); (-r, -s; (v_1, f_1))\rangle.$$

(ii) Prepare an ancillary qubit a in the state $|+_x\rangle_a = \frac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)$ and use this qubit to perform the controlled unitary operation $\wedge_1\left(X_e^{-r} Z_e^{s}\right) = |0\rangle_a\langle 0| \otimes \mathbf{1}_d + |1\rangle_a\langle 1| \otimes X_e^{-r} Z_e^{s}$ (with $(r, s) \neq (0, 0)$) on the qudit residing on the edge $e = [v_2, v_0]$. Measure the ancilla in the $\hat{x}$ basis and record the result $m = \pm 1$. The resultant state is $|\Psi(2)\rangle = \frac{1}{\sqrt{2}}\left(|\Psi(1)\rangle + (-1)^m X_e^{-r} Z_e^{s} |\Psi(1)\rangle\right)$, where

$$X_e^{-r} Z_e^{s} |\Psi(1)\rangle = |(a, b; (v_3, f_3)); (-a, -b; (v_4, f_4))\rangle \\ |(r, s; (v_2, f_2)); (-r, -s; (v_1, f_1))\rangle$$

is orthogonal to $|\Psi(1)\rangle$.

(iii) Use a sequence of local spin operations to drag the dyon at location (v_2, f_2) to the location (v_5, f_5). These operations should be done adiabatically, i.e. they should be done using localized control fields with frequency components much smaller than the minimum gap energy ΔE. In this way no new particles will be created, only the component of the wavefunction with the dyon located at (v_2, f_2) will be changed. Instead of using control fields to perform a discrete sequence of local spin operations, it is more robust

to slowly decrease the values of U and h while turning on generators for the local spin operations on the vertices and faces in the path from (v_2, f_2) to (v_5, f_5) so that it is energetically favourable for the dyon to follow this path. The resultant state is: $|\Psi(3)\rangle = \frac{1}{\sqrt{2}}(|\Psi(1)\rangle + (-1)^m|\Psi'\rangle)$, where

$$|\Psi'\rangle = |(a, b; (v_3, f_3)); (-a, -b; (v_4, f_4))\rangle \\ |(r, s; (v_5, f_5)); (-r, -s; (v_1, f_1))\rangle.$$

(iv) Braid the dyon $(a, b; (v_3, f_3))$ in a counterclockwise sense around the location (v_5, f_5) such that it returns to location (v_3, f_3). The state is now: $|\Psi(4)\rangle = \frac{1}{\sqrt{2}}(|\Psi(1)\rangle + (-1)^m \xi^{(sa+rb)}|\Psi'\rangle)$.

(v) Perform the inverse of the operations in step (iii), again ensuring that no new quasiparticles are created during the process. The resulting state is: $|\Psi(5)\rangle = \frac{1}{\sqrt{2}}(|\Psi(1)\rangle + (-1)^m \xi^{(sa+rb)}\, \mathrm{e}^{\mathrm{i}\chi} X_e^{-r} Z_e^s|\Psi(1)\rangle)$, where we have included χ, the sum of dynamical and Berry's phases that may have accumulated during steps (ii)–iv.

(vi) Reprepare the ancilla in the state $|+_x\rangle_a$ and perform the controlled unitary operation $\wedge_1\big((-1)^m Z_e^{-s} X_e^r\big)$. Measure the qubit in the $\hat{x}$ basis. The expectation value is:

$$\langle\sigma_a^x\rangle_\tau = \tfrac{1}{2}(\cos(\chi + \phi_{\mathrm{top}}) + \delta_{2r,0}\delta_{2s,0}\cos(\chi + \phi_{\mathrm{top}} - \phi_{AB})),$$

where $\phi_{\mathrm{top}} = 2\pi(sa + rb)/d$ is the topological phase associated with the exchange statistics of the two dyons and $\phi_{AB} = 2\pi rs/d$ is the Aharanov–Bohm phase of the dyon (r, s).

(vii) Repeat steps (i)–(vi) but measure the ancilla in the $\hat{y}$ basis. The expectation value is

$$\langle\sigma_a^y\rangle_\tau = \tfrac{1}{2}(\sin(\chi + \phi_{\mathrm{top}}) - \delta_{2r,0}\delta_{2s,0}\sin(\chi + \phi_{\mathrm{top}} - \phi_{AB})).$$

(viii) Perform a similar experiment but this time using a trivial braiding operation, i.e. perform the steps in the order (i), (ii), (iv), (iii), (v), (vi), (vii) so that the braid is contractible. Then the expectation values are

$$\langle\sigma_a^x\rangle_{\mathbf{1}} = \tfrac{1}{2}(\cos\chi + \delta_{2r,0}\delta_{2s,0}\cos(\chi - \phi_{AB})), \\ \langle\sigma_a^y\rangle_{\mathbf{1}} = \tfrac{1}{2}(\sin\chi - \delta_{2r,0}\delta_{2s,0}\sin(\chi - \phi_{AB})).$$

(ix) Compute the topological phase ϕ_{top} from an ensemble average obtained by repeated measurements on identically prepared systems.

As a simple example, consider the computation of the mutual statistics of charge and a flux for $d = 2$. Setting $(r, s) = (0, 1)$ and $(a, b) = (1, 0)$, the expected measurement results are $\langle\sigma_a^x\rangle_\tau = \frac{1}{2}\cos(\chi + \phi_{\mathrm{top}})$, $\langle\sigma_a^y\rangle_\tau = 0$, $\langle\sigma_a^x\rangle_{\mathbf{1}} = \frac{1}{2}\cos\chi$, $\langle\sigma_a^y\rangle_{\mathbf{1}} = 0$. If desired, the phase χ could be engineered to vary in a controlled manner over different trials in order to improve the visibility of the phase shift ϕ_{top}. For this case, the ancillary qubit is not necessary and one could apply the rotation operators $U_e = \mathrm{e}^{\mathrm{i}\pi Z_e/4}$ at step (ii) and $U_e^\dagger$ at step vi and measure $\langle Z_e\rangle$. For $d > 2$ it is always possible to choose the probe dyon such that $\delta_{2r,0}\delta_{2s,0} = 0$. In this case, ϕ_{top} is estimated by finding the closest solution to $\mathrm{e}^{\mathrm{i}\phi_{\mathrm{top}}} = \big(\langle\sigma_a^x\rangle_\tau + \mathrm{i}\langle\sigma_a^y\rangle_\tau\big)\big/\big(\langle\sigma_a^x\rangle_{\mathbf{1}} + \mathrm{i}\langle\sigma_a^y\rangle_{\mathbf{1}}\big)$.

There are three primary sources of error which could degrade the signal. First, after the creation of the quasiparticle pairs there is no energy penalty for moving the particles and in fact any local perturbing field acting on the surface can impart dynamics. If the particles or their associated antiparticles diffuse away from the intended braiding path then this will degrade the measured signal at the end of the protocol. The problem can be obviated by slightly decreasing the strength of the vertex and face constraints where the particles reside. For the particles with fixed locations this means reducing the local values of U and g there and for the moving particles this means changing the values of U and g along the braiding path. Doing

so creates a localizing potential which can confine the particles provided that the strength of the perturbing local field is small compared to the mass. Another source of error while traversing the braiding path is creating new unwanted quasiparticles. This can be avoided by performing local spin operations adiabatically with respect to the mass gap. Finally, there can be errors in applying the two controlled rotation operators between the ancilla qubit and its target edge qudit. Because these are controlled operations, in principle they as well as the final measurement can be done fault tolerantly.

7. Conclusions

We have proven the existence of a microscopic spin model that provides for topologically protected qudit encodings. This model describes a $\mathbb{Z}/d\mathbb{Z}$ gauge theory with Abelian charge/flux dyons as excitations. The construction is quite general, allowing for arbitrary cellulations of an orientable surface and encoding qudits with any finite number of levels. Suggested adaptations to the standard spin models using ancilla for in place stabilizer checks could prove advantageous in any physical implementation of such codes. Moreover, by performing local spin operations along a braiding path, the anyonic statistical phases can be measured with an ancillary particle. Recent interference experiments of a different kind have successfully measured anyonic statistics in specific fractional quantum Hall states [4]. The mechanism described here complements that approach to provide a novel set of tools for probing topological order in spin lattices.

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